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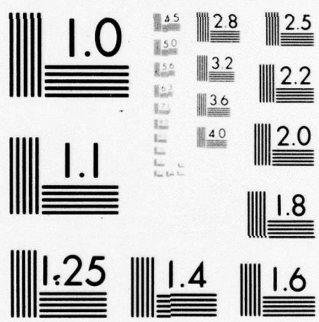

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by
WILLIAM S. JEWELL

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A CURIOUS RENEWAL PROCESS AVERAGE

by

William S. Jewell
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

JULY 1978 ORC 78-12

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ABSTRACT

The lifetimes of a renewal process observed during a fixed interval $(0,t]$ are smaller, on the average, than the process mean lifetime; it is shown that the mean observed lifetime has a particularly simple form.

A CURIOUS RENEWAL PROCESS AVERAGE

by

William S. Jewell

1. INTRODUCTION

Consider an ordinary renewal process, in which the nonnegative intervals $\{X_1, X_2, \ldots\}$ are independent random variables with common distribution F, and density f. The epoch of the n^{th} renewal, $Y_n = \sum_{i=1}^{n} X_i$, has density f^{n*}; the mean epoch is clearly at $nE\{X\}$.

If we fix an interval $(0,t]$, the random number of renewals in that interval, $N(t)$, has discrete density

$$P_n(t) = \Pr\{N(t) = n\} = \int_0^t P_{n-1}(t - x)dF(x) = F^{n*}(t) - F^{(n+1)*}(t) .$$

The mean number of renewals, $M(t)$, has a special role in renewal theory; it is given by the integral equation

$$M(t) = F(t) + \int_0^t M(t - x)dF(x) .$$

One interesting well-known "Waldian" formula concerns the next following renewal after time t; it occurs at mean value

$$E\{Y_{N(t)+1}\} = E\{X\} \cdot [M(t) + 1] .$$

However, simple results about $Y_{N(t)}$ do not seem to be obtainable; for example:

$$E\{Y_{N(t)} \mid N(t) > 0\} = \frac{\int_0^t [1 - F(t - x)]dM(x)}{F(t)} .$$

2. THE AVERAGE COMPLETED INTERVAL

However, the mean value of the completed intervals in $(0,t]$ has an unexpected simple form; this was discovered while investigating biases in renewal testing [1].

Theorem:

$$E\left\{\frac{Y_{N(t)}}{N(t)} \mid N(t) > 0\right\} = E\{X_1 \mid X_1 \le t\} = \frac{\int_0^t x dF(x)}{F(t)} = t - \int_0^t \left[\frac{F(x)}{F(t)}\right] dx \ .$$

A direct proof uses Laplace transforms on the numerator of:

$$E\left\{\frac{Y_{N(t)}}{N(t)} \mid N(t) > 0\right\} = \frac{\sum_{n=1}^{\infty} \frac{1}{n} \int_0^t [y f^{n*}(y)][1 - F(t - y)] dy}{F(t)} \ .$$

First, the transforms of the two terms in square brackets in the numerator are found in terms of the transform of f, and then multiplied, as the integral is a convolution; a term in n is created, fortuitously cancelling $1/n$, and the sum introduces further cancellation. The resulting expression is then recognized as the transform of $\int_0^t x dF(x)$.

A simple proof that perhaps explains the theorem better was suggested by H. Gerber, University of Michigan:

(1) It is easily verified that, given $N(t) = n > 0$,

$X_1, X_1, \ldots, X_n$ are *exchangeable random variables*;

(2) Therefore, given $N(t) = n > 0$, $E\{Y_n/n\} = E\{X_1\}$;

(3) But, cases in which $n = 1,2, \ldots$ are exactly those cases

in which $X_1 \leq t$. Q.E.D.

The average completed interval is approximately $t/2$ for small values of t , and increases monotonically to $E\{X_1\}$, usually slowly. For example, if f is exponential with parameter λ ,

$$E\{X_1 \mid X_1 \leq t\} = \frac{1}{\lambda}\left[\frac{1 - (1 + \lambda t)e^{-\lambda t}}{1 - e^{-\lambda t}}\right].$$

Unfortunately, the simplicity of the above result does not generalize to other functions of $W(t) = Y_{N(t)}/N(t)$, $(N(t) > 0)$. For example, the density of $W = w$ consists of different sums of convolutions over different intervals. The exponential case,

$$\lambda p(w) = \frac{\displaystyle\sum_{n=1}^{[t/u]} \frac{n^n(\lambda w)^{n-1}}{(n-1)!}}{(e^{\lambda t} - 1)} ,$$

is a real wonder, being level over $(t/2,t]$, linear over $(t/3,t/2]$, quadratic over $(t/4,t/3]$, etc. Figure 1 attempts to show this behavior for $\lambda t = 2$.

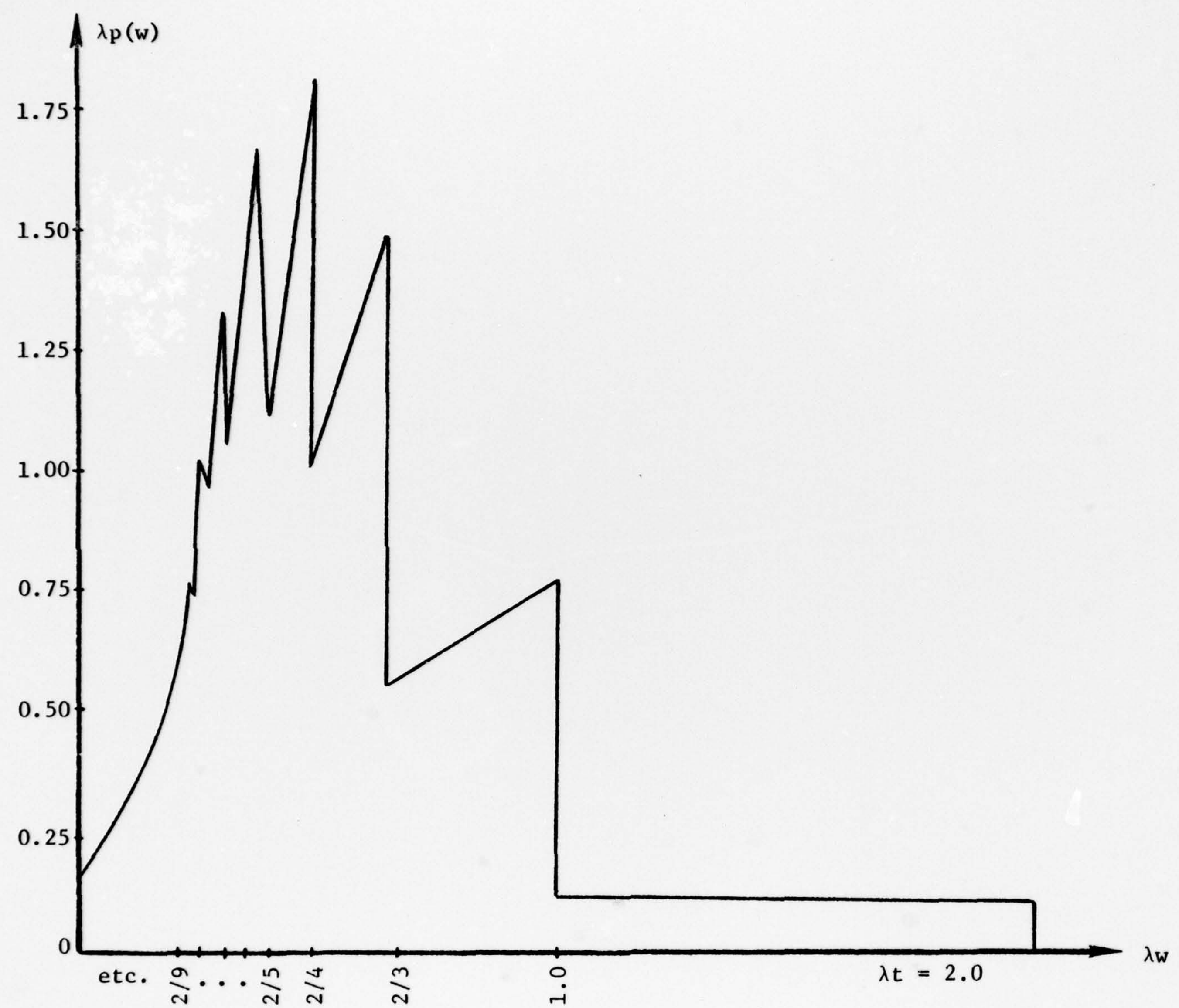

FIGURE 1

DENSITY OF AVERAGE COMPLETED INTERVALS IN $(0,t]$ FOR EXPONENTIAL INTERVALS WITH PARAMETER λ $(\lambda t = 2)$

REFERENCE

[1] Jewell, W. S., "'Reliability Growth' as an Artifact of Renewal Testing," ORC 78-9, Operations Research Center, University of California, Berkeley, (June 1978).